The Hermite Transform in Quaternionic Analysis

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1. Introduction

The conventional way of describing an image is in terms of its canonical pixel-based representation. Other image description techniques are based on image transformations. Such an image transformation converts a canonical image representation into a representation in which specific properties of an image are described more explicitly. In most transformations, images are locally approximated within a window by a linear combination of a number of a priori selected patterns. The coefficients of such a decomposition then provide the desired image representation.

The Hermite transform is an image transformation technique introduced by Martens in [6]; it uses overlapping Gaussian windows and projects images locally onto a basis of orthogonal polynomials.
As the analysis filters needed for the Hermite transform are derivatives of Gaussians, Hermite analysis is in close agreement with the information analysis carried out by the human visual system. It has been shown by Hubert and Wiesel (see [4]) that the visual cortex contains receptive fields with different profiles and orientations. According to the Gaussian derivative theory (see [9]), the shapes of these fields can be approximated quite well by the derivatives of two-dimensional Gaussians. The Hermite transform thus models the information analysis carried out by the retinal and cortical visual receptive fields. Because receptive fields occur in varying size, each field is suited for detecting the presence of a specific spatial frequency. With a two dimensional Hermite transform, field sizes can be modelled by varying the standard deviation of the Gaussian envelope, while orientation selectivity can be obtained by rotation of the Hermite filters.
So one can say that in image representation and compression theory, the Hermite transform is located somewhere between the conventional transform coding techniques and second-generation coding techniques (see [5]). The basic principle for these second-generation coding techniques is that image

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analysis should be related more closely to the way in which the visual system analyzes visual information and image representations should be tuned to the explicit description of perceptually important image structures (e.g. edge contours and textures).

Up to now, Hermite transforms have been used in applications such as image compression, image deblurring, noise reduction and estimation of perceived noise en blur.
Moreover, in [3] it is shown that the Hermite transform is also very efficient in image analysis (detection and extraction of image features).

In this paper we construct a new higher dimensional Hermite transform within the framework of Quaternionic Analysis (section 4). The building blocks for this construction are the Clifford-Hermite polynomials introduced in [7] in the framework of Clifford Analysis, rewritten in terms of Quaternionic analysis.
Furthermore, we compare this newly introduced Hermite transform with the $\mathbb{H}$-Hermite Continuous Wavelet transform (section 5 and 6).
Finally the developed three dimensional filter functions are tested with traditional scalar benchmark signals upon their selectivity at detecting pointwise singularities (section 7).

2. The classical Hermite transform

The Hermite transform was introduced in [6] as a signal expansion technique in which a signal is windowed at equidistant positions and is locally described as a weighted sum of polynomials. In this section we give a brief overview of the one dimensional classical Hermite transform; by a tensorial approach it is generalized to higher dimension in a straightforward manner.

The first step in the Hermite transform is to localize the original signal $L(x)$ by multiplying it by a Gaussian window function

$$\widetilde{V}^{\sigma}(x) = \frac{1}{\sqrt{\sqrt{\pi}\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

A complete description of the signal $L(x)$ requires that the localization process is repeated at a sufficient number of window positions, the spacing between the windows being chosen equidistant. In this way the following expansion of the original signal $L(x)$ is obtained:

$$L(x) = \frac{1}{\widetilde{W}^{\sigma}(x)} \sum_{k=-\infty}^{+\infty} L(x)\widetilde{V}^{\sigma}(x-kT), \tag{1}$$

with

$$\widetilde{W}^{\sigma}(x) = \sum_{k=-\infty}^{+\infty} \widetilde{V}^{\sigma}(x - kT).$$

the so-called weight function, which is positive for all x.

The second step consists of decomposing the localized signal $L(x)\widetilde{V}^{\sigma}(x - kT)$ into a series of orthogonal functions $K_n^{\sigma}(x) = \widetilde{V}^{\sigma}(x)G_n^{\sigma}(x)$:

$$\widetilde{V}^{\sigma}(x - kT)L(x) = \sum_{n=0}^{\infty} L_n^{\sigma}(kT)K_n^{\sigma}(x - kT), \tag{2}$$

with $G_n^{\sigma}(x)$ the uniquely determined polynomials which are orthonormal with respect to $(\widetilde{V}^{\sigma}(x))^2$.
The coefficients $L_n^{\sigma}(kT)$ in the above decomposition of the localized signal are called the *Hermite coefficients* and are given by

$$L_n^{\sigma}(kT) = \int_{-\infty}^{+\infty} L(x)G_n^{\sigma}(x - kT)(\widetilde{V}^{\sigma}(x - kT))^2 dx.$$

This defining relation of the Hermite coefficients can be rewritten as the convolution of the original signal $L(x)$ with the so-called filter functions $D_n^{\sigma}(x)$, followed by a downsampling by a factor T.
These filter functions $D_n^{\sigma}(x)$ can be expressed as the derivatives of a Gaussian:

$$D_n^{\sigma}(x) = \frac{\sigma^n}{\sqrt{2^n n!}} \frac{d^n}{dx^n} \left(\frac{1}{\sigma\sqrt{\pi}} \exp\left(-\frac{x^2}{\sigma^2} \right) \right).$$

Combining the decompositions (1) and (2), we get the expansion of the complete signal into the pattern functions Q_n^{σ}:

$$L(x) = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{+\infty} L_n^{\sigma}(kT)Q_n^{\sigma}(x - kT),$$

with

$$Q_n^{\sigma}(x) = \frac{K_n^{\sigma}(x)}{\widetilde{W}^{\sigma}(x)}.$$

The mapping from the original signal $L(x)$ to the Hermite coefficients $L_n^{\sigma}(kT)$ is called the forward Hermite transform. The signal reconstruction from the Hermite coefficients is called the inverse Hermite transform.

3. Quaternionic Analysis

Quaternionic Analysis (see [8]) offers a function theory which is a three or four dimensional analogue of the theory of the holomorphic functions of one complex variable.

We consider functions defined in $\mathbb{R}^3$ and taking values in the quaternions $\mathbb{H}$ spanned by $(1, e_1, e_2, e_{12})$ where

$$\begin{array}{rcl} e_1^2 = e_2^2 & = & e_{12}^2 = -1 \\ e_1e_2 = e_{12} = -e_2e_1 \ ; \ e_2e_{12} = & e_1 & = -e_{12}e_2 \ ; \ e_{12}e_1 = e_2 = -e_1e_{12} \end{array}$$

and 1 is the identity element.

Conjugation is defined as the anti-involution for which

$$\overline{e_1} = -e_1; \quad \overline{e_2} = -e_2; \quad \overline{e_{12}} = -e_{12},$$

with the additional rule

$$\bar{i} = -i$$

where complex coefficients are involved.

The Euclidean space $\mathbb{R}^3$ is embedded in the pure quaternions $\mathbb{H}^{(p)}$ by identifying (x, y, z) with the pure quaternion $\underline{x}$ given by

$$\underline{x} = xe_1 + ye_2 + ze_{12}.$$

The non commutative product of two pure quaternions $\underline{x}$ and $\underline{x'} = x'e_1 + y'e_2 + z'e_{12}$ splits up into a scalar part and a pure quaternion part:

$$\underline{x}\ \underline{x'} = -xx' - yy' - zz' + (yz' - y'z)e_1 + (x'z - xz')e_2 + (xy' - x'y)e_{12}. \quad (3)$$

The scalar product of the two pure quaternions $\underline{x}$ and $\underline{x'}$ is defined as to be

$$< \underline{x}, \underline{x'} >= \mathrm{Re}(\overline{\underline{x}}\ \underline{x'}) = xx' + yy' + zz',$$

where $\mathrm{Re}(\overline{\underline{x}}\ \underline{x'})$ stands for the scalar part of the quaternion product $\overline{\underline{x}}\ \underline{x'}$. Consequently, the product (3) of the two pure quaternions $\underline{x}$ and $\underline{x'}$ can be written as

$$\underline{x}\ \underline{x'} = - < \underline{x}, \underline{x'} > + \underline{x} \times \underline{x'},$$

with

$$\underline{x} \times \underline{x'} = (yz' - y'z)e_1 + (x'z - xz')e_2 + (xy' - x'y)e_{12}.$$

the vector product of $\underline{x}$ and $\underline{x'}$.

In particular we have that

$$\underline{x}^2 = -\overline{\underline{x}}\ \underline{x} = - < \underline{x}, \underline{x} >= -|\underline{x}|^2.$$

This implies that each non-zero pure quaternion $\underline{x}$ does have a multiplicative inverse:

$$\underline{x}^{-1} = \frac{\overline{\underline{x}}}{|\underline{x}|^2}.$$

A fundamental operator is

$$\underline{\partial} = \partial_x e_1 + \partial_y e_2 + \partial_z e_{12},$$

which splits the Laplace operator in $\mathbb{R}^3$:

$$\underline{\partial}^2 = -(\partial_x^2 + \partial_y^2 + \partial_z^2) = -\Delta.$$

Nullsolutions of this operator, i.e. functions satisfying

$$\underline{\partial} F = 0,$$

are called left monogenic functions.

To illustrate this concept of monogenicity, let us associate with the function F the vector field

$$\vec{F} = (f, g, h) = f\vec{e_x} + g\vec{e_y} + h\vec{e_z}$$

with $(\vec{e_x}, \vec{e_y}, \vec{e_z})$ a right-handed orthonormal frame in $\mathbb{R}^3$. Then the left monogenicity of the function F in the open region Ω of $\mathbb{R}^3$ is equivalent with the so-called Riesz system for the vector field $\vec{F}$:

$$\left\{ \begin{array}{lcl} \mathrm{div}\vec{F} & = & 0 \\ \mathrm{curl}\vec{F} & = & 0 \end{array} \right.$$

in Ω.

In the sequel the Fourier transform of $F(\underline{x})$ will be denoted by $\widehat{F(\underline{x})}(\underline{u})$:

$$\widehat{F(\underline{x})}(\underline{u}) = \int_{\mathbb{R}^3} \exp\left(-i < \underline{x}, \underline{u} >\right) F(\underline{x}) dV(\underline{x}),$$

where $dV(\underline{x})$ stands for the Lebesgue measure on $\mathbb{R}^3$.

4. The Quaternionic-Hermite transform

4.1 The Quaternionic-Hermite polynomials

Analogous to the Clifford-Hermite polynomials (see [7]), we define the $\mathbb{H}$-Hermite polynomials by the relation:

$$H_n(\underline{x}) = (-1)^n \exp\left(\frac{|\underline{x}|^2}{2}\right)\underline{\partial}^n \exp\left(-\frac{|\underline{x}|^2}{2}\right), \quad n = 0, 1, 2, \ldots$$

A straightforward calculation yields

$$\begin{aligned}
H_0(\underline{x}) &= 1 \\
H_1(\underline{x}) &= \underline{x} = r\underline{\omega} \\
H_2(\underline{x}) &= \underline{x}^2 + 3 = -r^2 + 3 \\
H_3(\underline{x}) &= \underline{x}^3 + 5\underline{x} = (-r^3 + 5r)\underline{\omega} \\
H_4(\underline{x}) &= \underline{x}^4 + 10\underline{x}^2 + 15 = r^4 - 10r^2 + 15,
\end{aligned}$$

where we have introduced spherical co-ordinates

$$\begin{aligned}
\underline{x} = r\underline{\omega} \quad &; \quad r = |\underline{x}|; \\
\underline{\omega} = \sin\theta\cos\phi e_1 + \sin\theta\sin\phi e_2 + \cos\theta e_{12} \in S^2 \quad &, \quad \theta \in [0,\pi],\ \phi \in [0, 2\pi]
\end{aligned}$$

where S^2 denotes the unit sphere in $\mathbb{R}^3$.

Note that $H_k(\underline{x})$ is a polynomial of degree k in the variable $\underline{x}$, that $H_{2k}(\underline{x})$ only contains even powers, while $H_{2k+1}(\underline{x})$ only contains odd ones. Consequently $H_{2k}(\underline{x})$ is real valued, while $H_{2k+1}(\underline{x})$ is $\mathbb{H}^{(p)}$-valued.

The $\mathbb{H}$-Hermite polynomials satisfy the recurrence relation

$$H_{n+1}(\underline{x}) = (\underline{x} - \underline{\partial})H_n(\underline{x})$$

and the orthogonality relation

$$\int_{\mathbb{R}^3} \exp\left(-\frac{|\underline{x}|^2}{2}\right)\overline{H_k}(\underline{x})H_l(\underline{x})dV(\underline{x}) = \gamma_k\delta_{k,l}, \tag{4}$$

with

$$\gamma_{2p} = \frac{2^{2p+3/2}p!\pi^{3/2}\Gamma(\frac{3}{2}+p)}{\Gamma(\frac{3}{2})}$$

and

$$\gamma_{2p+1} = \frac{2^{2p+3/2+1}p!\pi^{3/2}\Gamma(\frac{3}{2}+p+1)}{\Gamma(\frac{3}{2})}.$$

4.2 The Quaternionic-Hermite transform

In this subsection we develop a new three dimensional Hermite transform in the framework of Quaternionic Analysis, which we call the $\mathbb{H}$-Hermite transform.

Using the real valued Gaussian window function

$$V^\sigma(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2\sigma^2}\right) = \exp\left(-\frac{x^2+y^2+z^2}{2\sigma^2}\right),$$

we get the following decomposition of the original $\mathbb{H}$-valued signal $L(\underline{x})$:

$$L(\underline{x}) = \frac{1}{W^\sigma(\underline{x})} \sum_{\underline{p}\in P} L(\underline{x}) V^\sigma(\underline{x}-\underline{p}) \tag{5}$$

where

$$W^\sigma(\underline{x}) = \sum_{\underline{p}\in P} V^\sigma(\underline{x}-\underline{p})$$

is the positive scalar-valued weight function and P a sampling grid in $\mathbb{R}^3$.

For the decomposition of the localized signal $L(\underline{x})V^\sigma(\underline{x}-\underline{p})$, we use the $\mathbb{H}$-Hermite polynomials (see previous subsection) which satisfy the orthogonality relation (see (4))

$$\int_{\mathbb{R}^3} \overline{H_n}\left(\frac{\sqrt{2}}{\sigma}\underline{x}\right) H_{n'}\left(\frac{\sqrt{2}}{\sigma}\underline{x}\right) (V^\sigma(\underline{x}))^2 dV(\underline{x}) = \frac{\sigma^3}{2^{3/2}}\gamma_n \delta_{n,n'}.$$

Now we introduce the alternatively scalar- or $\mathbb{H}^{(p)}$-valued functions $K_n^\sigma(\underline{x}) = V^\sigma(\underline{x}) H_n(\frac{\sqrt{2}}{\sigma}\underline{x})$ which satisfy the orthogonality relation

$$\int_{\mathbb{R}^3} \overline{K_n^\sigma}(\underline{x}) K_{n'}^\sigma(\underline{x}) dV(\underline{x}) = \frac{\sigma^3}{2^{3/2}}\gamma_n \delta_{n,n'}.$$

Under very general conditions for the original signal $L(\underline{x})$, the above orthogonality relation leads to the following decomposition of the localized signal into the orthogonal functions $K_n^\sigma(\underline{x})$:

$$V^\sigma(\underline{x}-\underline{p}) L(\underline{x}) = \sum_{j=0}^{\infty} L_j^\sigma(\underline{p}) \overline{K_j^\sigma}(\underline{x}-\underline{p}). \tag{6}$$

Here we have put

$$
\begin{aligned}
L_n^\sigma(\underline{p}) &= \frac{2^{3/2}}{\sigma^3\gamma_n}\int_{\mathbb{R}^3} L(\underline{x}) H_n\left(\frac{\sqrt{2}(\underline{x}-\underline{p})}{\sigma}\right)(V^\sigma(\underline{x}-\underline{p}))^2 dV(\underline{x}) \\
&= \frac{2^{3/2}}{\sigma^3\gamma_n}\int_{\mathbb{R}^3} L(\underline{x}) K_n^\sigma(\underline{x}-\underline{p}) V^\sigma(\underline{x}-\underline{p}) dV(\underline{x}).
\end{aligned}
$$

We call $L_n^\sigma(\underline{p})$ the $\mathbb{H}$-Hermite coefficients; they are $\mathbb{H}$-valued.

Combining (5) and (6), we obtain the decomposition of the complete signal:

$$
L(\underline{x}) = \sum_{j=0}^{\infty}\sum_{\underline{p}\in P} L_j^\sigma(\underline{p}) Q_j^\sigma(\underline{x}-\underline{p}),
$$

where we have introduced the pattern functions

$$
Q_j^\sigma(\underline{x}) = \frac{\overline{K_j^\sigma}(\underline{x})}{W^\sigma(\underline{x})} = \frac{\overline{H_j}(\frac{\sqrt{2}}{\sigma}\underline{x})V^\sigma(\underline{x})}{W^\sigma(\underline{x})}.
$$

Note that the $\mathbb{H}$-Hermite coefficients may be expressed as the convolution of the original signal with the $\mathbb{H}$-Hermite filter functions

$$
\begin{aligned}
D_n^\sigma(\underline{x}) &= \frac{2^{3/2}}{\sigma^3\gamma_n} H_n\left(-\frac{\sqrt{2}}{\sigma}\underline{x}\right)(V^\sigma(-\underline{x}))^2 \\
&= \frac{(-1)^n 2^{3/2}}{\sigma^3\gamma_n} H_n\left(\frac{\sqrt{2}}{\sigma}\underline{x}\right)\exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \\
&= \frac{2^{(3-n)/2}\sigma^{n-3}}{\gamma_n}\underline{\partial}^n\left(\exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right)\right),
\end{aligned}
$$

with Fourier transform in spherical co-ordinates $\underline{u} = \rho\underline{\xi}$; $\underline{\xi} \in S^2$ given by

$$
\widehat{D_n^\sigma}(\underline{u}) = \frac{(i\sigma)^n}{\gamma_n 2^{n/2}}(2\pi)^{3/2}\underline{\xi}^n\rho^n \exp\left(-\frac{\sigma^2\rho^2}{4}\right).
$$

These filter functions have the property of being polar separable, i.e. their Fourier transform is expressed as the product of a spatial frequency tuning function and an orientation tuning function. Daugman has demonstrated the importance of polar separable filters in [2] .

Note that if the original signal is real-valued, the $\mathbb{H}$-Hermite coefficients are alternatively scalar- or $\mathbb{H}^{(p)}$-valued. The case of a $\mathbb{H}^{(p)}$-valued signal is treated in the next subsection.

4.3 Analysis of a pure quaternion-valued signal

As $H_{2N+1}(\underline{x}) = B_N(|\underline{x}|)\underline{x}$, with $B_N(|\underline{x}|)$ a scalar-valued polynomial of degree N in the variable $|\underline{x}|^2$, the $\mathbb{H}$-Hermite filter functions of odd order take the following form:

$$\begin{aligned} D^{\sigma}_{2N+1}(\underline{x}) &= -\frac{2^{3/2}}{\sigma^3 \gamma_{2N+1}} H_{2N+1}\left(\frac{\sqrt{2}}{\sigma}\underline{x}\right) \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \\ &= -\frac{4}{\sigma^4 \gamma_{2N+1}} \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) B_N\left(\frac{\sqrt{2}}{\sigma}|\underline{x}|\right)\underline{x} \\ &= \tilde{D}^{\sigma}_{2N+1}(\underline{x})\underline{x}, \end{aligned}$$

with $\tilde{D}^{\sigma}_{2N+1}(\underline{x})$ the scalar factor in $D^{\sigma}_{2N+1}(\underline{x})$ depending upon $|\underline{x}|^2$.

The corresponding $\mathbb{H}$-Hermite coefficient of the $\mathbb{H}^{(p)}$-valued signal

$$L(\underline{x}) = f(\underline{x})e_1 + g(\underline{x})e_2 + h(\underline{x})e_{12}; \quad f, g, h : \mathbb{R}^3 \to \mathbb{R},$$

takes the form:

$$\begin{aligned} L^{\sigma}_{2N+1}(\underline{p}) &= \int_{\mathbb{R}^3} L(\underline{x}) D^{\sigma}_{2N+1}(\underline{p}-\underline{x}) dV(\underline{x}) \\ &= \int_{\mathbb{R}^3} \tilde{D}^{\sigma}_{2N+1}(\underline{p}-\underline{x}) L(\underline{x})(\underline{p}-\underline{x}) dV(\underline{x}) \\ &= -\int_{\mathbb{R}^3} \tilde{D}^{\sigma}_{2N+1}(\underline{p}-\underline{x}) < L(\underline{x}), \underline{p}-\underline{x} > dV(\underline{x}) \\ &\quad + \int_{\mathbb{R}^3} \tilde{D}^{\sigma}_{2N+1}(\underline{p}-\underline{x}) [L(\underline{x}) \times (\underline{p}-\underline{x})] \, dV(\underline{x}) \\ &= L^{\sigma;0}_{2N+1}(\underline{p}) + L^{\sigma;1}_{2N+1}(\underline{p}). \end{aligned}$$

So, as a consequence of the structure of the product of two pure quaternions, the $\mathbb{H}$-Hermite coefficients of odd order of a $\mathbb{H}^{(p)}$-valued signal split up into a scalar part $L^{\sigma;0}_{2N+1}(\underline{p})$ and a pure quaternion part $L^{\sigma;1}_{2N+1}(\underline{p})$.
Moreover, in the integral defining the scalar part there are no contributions in the direction orthogonal to the signal, whereas in the integral defining the pure quaternion part there are no contributions in the direction of the signal itself.

We end this subsection with the remark that the $\mathbb{H}$-Hermite coefficients of even order of a $\mathbb{H}^{(p)}$-valued signal are $\mathbb{H}^{(p)}$-valued.

5. The Quaternionic-Hermite Continuous Wavelet Transform

5.1 The Continuous Wavelet Transform

The continuous wavelet transform (CWT) is a signal analysis technique suitable for non-stationary, inhomogeneous signals for which Fourier analysis is inadequate (see e.g. [1]).
In the one dimensional case, it is given by the integral transform

$$L(x) \to L(a,b) = \int_{-\infty}^{+\infty} \overline{g^{a,b}}(x) L(x) dx.$$

The kernel function of this integral transform is the dilate translate of a so-called mother wavelet g:

$$g^{a,b}(x) = \frac{1}{\sqrt{a}} g\left(\frac{x-b}{a}\right), \qquad a > 0,\ b \in \mathbb{R},$$

where the parameter b indicates the position of the wavelet, while the parameter a governs its frequency. The analyzing wavelet function g is a quite arbitrary L_2-function which is well localized both in the time domain and in the frequency domain. Moreover it has to satisfy the so-called admissibility condition:

$$C_g := \int_{-\infty}^{+\infty} \frac{|\widehat{g}(\xi)|^2}{|\xi|} d\xi < +\infty,$$

where $\widehat{g}$ denotes the Fourier transform of g. The constant C_g is called the admissibility constant. In the case where g is also L_1, this admissibility condition implies that g has mean value zero, is oscillating and decays to zero at infinity. These properties explain the qualification as "wavelet" of this function g.
The CWT may be extended to higher dimensions while enjoying the same properties as in the one dimensional case.

5.2 Quaternionic-Hermite wavelets

The orthogonality relation (4) implies that for $n > 0$

$$\int_{\mathbb{R}^3} \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_n(\underline{x}) dV(\underline{x}) = 0.$$

In terms of wavelet theory this means that the $L_1 \cap L_2$-functions

$$\begin{aligned}\psi_n(\underline{x}) &= \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_n(\underline{x}) \\ &= (-1)^n \underline{\partial}^n \exp\left(-\frac{|\underline{x}|^2}{2}\right)\end{aligned}$$

have zero momentum and are a good type of basic wavelet kernel functions in $\mathbb{R}^3$ if at least they satisfy an appropriate admissibility condition (see subsection 5.3). We call them the $\mathbb{H}$-Hermite wavelets.

The $\mathbb{H}$-Hermite wavelets are alternatively scalar- or $\mathbb{H}^{(p)}$-valued and their Fourier transform is given by

$$\widehat{\psi_n}(\underline{u}) = (2\pi)^{3/2}(-i)^n \underline{u}^n \exp\left(-\frac{|\underline{u}|^2}{2}\right).$$

Moreover it follows from the orthogonality relation (4) that the $\mathbb{H}$-Hermite wavelet ψ_n has vanishing moments up to order $n-1$:

$$\int_{\mathbb{R}^3} \underline{x}^j \psi_n(\underline{x}) dV(\underline{x}) = 0; \quad j = 0, \ldots, n-1.$$

As the capacity of the wavelets for detecting singularities in a signal is related to their number of vanishing moments, this last result means that the $\mathbb{H}$-Hermite wavelet ψ_n is particularly appropriate for pointwise signal analysis, whereby the corresponding CWT will filter out polynomial behaviour of the signal up to degree $n-1$.

5.3 The Quaternionic-Hermite Continuous Wavelet Transform

In order to introduce the corresponding $\mathbb{H}$-Hermite CWT, we consider, still for $n > 0$, the continuous family of wavelets

$$\psi_n^{a,\underline{b}}(\underline{x}) = \frac{1}{a^{3/2}} \psi_n\left(\frac{\underline{x}-\underline{b}}{a}\right);$$

where $a \in \mathbb{R}_+$ and $\underline{b} \in \mathbb{R}^3$.
They originate from the basic wavelet function ψ_n by dilation and translation. Their Fourier transform is given by

$$\widehat{\psi}_n^{a,\underline{b}}(\underline{u}) = a^{3/2} \exp\left(-i < \underline{b}, \underline{u} >\right) \widehat{\psi}_n(a\underline{u}).$$

The $\mathbb{H}$-Hermite CWT applies to functions $F \in L_2(\mathbb{R}^3)$ by

$$\begin{aligned} T_n^{\mathbb{H}} : F(\underline{x}) \to F_n(a, \underline{b}) &= \int_{\mathbb{R}^3} \overline{\psi_n^{a,\underline{b}}}(\underline{x}) F(\underline{x}) dV(\underline{x}) \\ &= \frac{1}{a^{3/2}} \int_{\mathbb{R}^3} \exp\left(-\frac{|\underline{x}-\underline{b}|^2}{2a^2}\right) \overline{H_n} \left(\frac{\underline{x}-\underline{b}}{a}\right) F(\underline{x}) dV(\underline{x}). \end{aligned} \tag{7}$$

This definition can be rewritten in terms of the Fourier transform as

$$\begin{aligned} F_n(a, \underline{b}) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \overline{\widehat{\psi_n^{a,\underline{b}}}}(\underline{u}) \widehat{F}(\underline{u}) dV(\underline{u}) \\ &= \frac{a^{3/2}}{(2\pi)^3} \int_{\mathbb{R}^3} \exp\left(i < \underline{b}, \underline{u} >\right) \overline{\widehat{\psi}_n}(a\underline{u}) \widehat{F}(\underline{u}) dV(\underline{u}) \\ &= \frac{a^{3/2}}{(2\pi)^3} \widehat{\overline{\widehat{\psi}_n}(a\underline{u}) \widehat{F}(\underline{u})}(-\underline{b}). \end{aligned}$$

It is clear that the $\mathbb{H}$-Hermite CWT will map $L_2(\mathbb{R}^3)$ into a weighted L_2-space on $\mathbb{R}_+ \times \mathbb{R}^3$ for some weight function still to be determined. This weight function has to be chosen in this way that the $\mathbb{H}$-Hermite CWT is an isometry, or in other words that the Parseval formula should hold.
Introducing the inner product

$$[F_n, G_n] = \frac{1}{C_n} \int_{\mathbb{R}^3} \int_0^{+\infty} \overline{F_n(a, \underline{b})} G_n(a, \underline{b}) \frac{da}{a^4} dV(\underline{b}),$$

we search for the constant C_n in order to have the Parseval formula

$$< F, G >= [F_n, G_n]$$

fulfilled.
We have consecutively

$$\begin{aligned} [F_n, G_n] &= \frac{C_n^{-1}}{(2\pi)^6} \int_{\mathbb{R}^3} \int_0^{+\infty} \overline{\widehat{\overline{\widehat{\psi}_n}(a\underline{u}) \widehat{F}(\underline{u})}(-\underline{b})} \widehat{\overline{\widehat{\psi}_n}(a\underline{u}) \widehat{G}(\underline{u})}(-\underline{b}) \frac{da}{a} dV(\underline{b}) \\ &= \frac{C_n^{-1}}{(2\pi)^3} \int_{\mathbb{R}^3} \int_0^{+\infty} \overline{\widehat{F}(\underline{u})} \widehat{\psi}_n(a\underline{u}) \overline{\widehat{\psi}_n}(a\underline{u}) \widehat{G}(\underline{u}) \frac{da}{a} dV(\underline{u}). \end{aligned}$$

If we put

$$\int_0^{+\infty} \widehat{\psi}_n(a\underline{u}) \overline{\widehat{\psi}_n}(a\underline{u}) \frac{da}{a} = C_n,$$

we get the desired result

$$[F_n, G_n] = \frac{1}{(2\pi)^3} < \widehat{F}, \widehat{G} >=< F, G > .$$

This means that if the functions ψ_n satisfy the relation defining the constant C_n, the $\mathbb{H}$-Hermite CWT will be an isometry between two L_2-spaces. This relation is called the admissibility condition for the $\mathbb{H}$-Hermite wavelets, and the constant C_n involved is the admissibility constant.
By means of the substitution

$$\underline{u} = \frac{r}{a}\underline{\omega}, \quad \underline{\omega} \in S^2$$

and taking into account that $\widehat{\psi}_n(\underline{\xi})\overline{\widehat{\psi}}_n(\underline{\xi})$ is $SO(3)$-invariant, the admissibility condition may be simplified to

$$\begin{aligned}
C_n &= \int_0^{+\infty} \widehat{\psi}_n(r\underline{\omega})\overline{\widehat{\psi}}_n(r\underline{\omega})\frac{dr}{r} \\
&= \frac{1}{A_2}\int_{\mathbb{R}^3} \frac{\widehat{\psi}_n(\underline{\xi})\overline{\widehat{\psi}}_n(\underline{\xi})}{|\underline{\xi}|^3} dV(\underline{\xi}) \\
&= \frac{(2\pi)^3}{A_2}\int_{\mathbb{R}^3} |\underline{\xi}|^{2n-3}\exp\left(-|\underline{\xi}|^2\right) dV(\underline{\xi}) \\
&= (2\pi)^3\frac{(n-1)!}{2} < +\infty.
\end{aligned}$$

The $\mathbb{H}$-Hermite CWT maps $L_2(\mathbb{R}^3)$ into $L_2(\mathbb{R}_+ \times \mathbb{R}^3, C_n^{-1}a^{-4})$ but it is by no means a surjection onto this space. So there is a lot of freedom in constructing inversion formulae. However from the above Parseval formula it follows that if $F \in L_2(\mathbb{R}^3)$ and $F_n(a,\underline{b}) = T_n^{\mathbb{H}}F(\underline{x})$ then:

$$\begin{aligned}
F(\underline{x}) &= \frac{1}{C_n}\int_{\mathbb{R}^3}\int_0^{+\infty} \psi_n^{a,\underline{b}}(\underline{x})F_n(a,\underline{b})\frac{da}{a^4}dV(\underline{b}) \qquad (8) \\
&= [\overline{\psi}_n^{a,\underline{b}}(\underline{x}), F_n(a,\underline{b})]
\end{aligned}$$

to hold weakly in $L_2(\mathbb{R}^3)$.
This means that the signal $F(\underline{x})$ may be reconstructed from its transform $F_n(a,\underline{b})$ or, in other words, that the $\mathbb{H}$-Hermite CWT decomposes the signal $F(\underline{x})$ in terms of the analyzing wavelets $\psi_n^{a,\underline{b}}(\underline{x})$ with coefficients $F_n(a,\underline{b})$.

6. Comparison of the Quaternionic-Hermite transform with the Quaternionic-Hermite Continuous Wavelet Transform

Putting $\sigma = \sqrt{2}\mu$ in the $\mathbb{H}$-Hermite coefficients of subsection 4.2

$$\begin{aligned} L_n^\sigma(\underline{p}) &= \int_{\mathbb{R}^3} L(\underline{x}) D_n^\sigma(\underline{p}-\underline{x}) dV(\underline{x}) \\ &= \frac{2^{3/2}}{\sigma^3\gamma_n}\int_{\mathbb{R}^3} L(\underline{x}) H_n\left(\frac{\sqrt{2}(\underline{x}-\underline{p})}{\sigma}\right)\exp\left(-\frac{|\underline{x}-\underline{p}|^2}{\sigma^2}\right) dV(\underline{x}), \end{aligned}$$

we obtain

$$L_n^{\sqrt{2}\mu}(\underline{p}) = \frac{1}{\mu^3\gamma_n}\int_{\mathbb{R}^3} L(\underline{x}) H_n\left(\frac{\underline{x}-\underline{p}}{\mu}\right)\exp\left(-\frac{|\underline{x}-\underline{p}|}{2\mu^2}\right) dV(\underline{x}). \qquad (9)$$

Comparing (7) and (9), the connection between the $\mathbb{H}$-Hermite transform and the $\mathbb{H}$-Hermite CWT is clear: σ plays the role of the dilation parameter a, $\underline{p}$ plays the role of the translation parameter $\underline{b}$ and the translated filter functions $D_n^\sigma(\underline{p}-\underline{x})$ play the role of the wavelets $\psi_n^{a,\underline{b}}$.

There are however some differences between the two transforms. The wavelet translation parameter $\underline{b}$ is continuous, whereas the parameter $\underline{p}$ in the Hermite transform is discrete. Furthermore, the reconstruction of the original signal in the Hermite transform

$$L(\underline{x}) = \sum_{j=0}^{\infty}\sum_{\underline{p}\in P} L_j^\sigma(\underline{p}) Q_j^\sigma(\underline{x}-\underline{p}) \qquad (10)$$

depends on σ, which is an important parameter for practical applications. In order for the finite Hermite transform to describe the signal adequately, σ must be properly selected. On the one hand, we want σ to be as large as possible because integrating over large areas improves the output signal-to-noise ratio as well as the efficiency of our signal representation. On the other hand, σ cannot be too large because then the signal cannot be described accurately by the first few terms in the Hermite expansion.

In the CWT, the quality of the signal reconstruction (8) depends on the mother wavelet $\psi_n(\underline{x})$; in other words it depends on the order of the derivative of the Gaussian. In the inverse Hermite transform (10) we sum over all these orders.

7. Benchmarking of a Quaternionic-Hermite filter function

In this section we demonstrate that the filter functions of the $\mathbb{H}$-Hermite transform are selective at detecting pointwise singularities of the signal.

We test this capacity of the filter function

$$\begin{aligned} D_1^\sigma(\underline{x}) &= \frac{2^{3/2}}{\sigma^3\gamma_1} H_1\left(-\frac{\sqrt{2}\underline{x}}{\sigma}\right)\exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \\ &= \frac{4}{\sigma^3\gamma_1}\left(-\frac{\underline{x}}{\sigma}\right)\exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right). \end{aligned}$$

The $\mathbb{H}$-Hermite coefficients of the signal $L(\underline{x})$ that correspond with this filter function are

$$\begin{aligned} L_1^\sigma(\underline{p}) &= \int_{\mathbb{R}^3} L(\underline{x}) D_1^\sigma(\underline{p}-\underline{x})dV(\underline{x}) \\ &= \frac{4}{\sigma^3\gamma_1}\int_{\mathbb{R}^3} L(\underline{x})\left(\frac{\underline{x}-\underline{p}}{\sigma}\right)\exp\left(-\frac{|\underline{x}-\underline{p}|^2}{\sigma^2}\right)dV(\underline{x}). \end{aligned}$$

The tests are carried out on scalar benchmark signals: the infinite rod, the semi-infinite rod and the rod of finite length.

First, the filter function is tested on the scalar benchmark signal consisting of an infinite rod, laying along the x-axis, modelled as

$$L(\underline{x}) = \delta(y)\delta(z),$$

where δ stands for the delta distribution.
Its $\mathbb{H}$-Hermite coefficients are

$$L_1^\sigma(\underline{p}) = -\frac{4\sqrt{\pi}}{\sigma^3\gamma_1}\exp\left(-\frac{p_2^2}{\sigma^2}\right)\exp\left(-\frac{p_3^2}{\sigma^2}\right)(p_2e_2 + p_3e_{12}).$$

We observe that the coefficients decay rapidly to zero for $(p_2, p_3) \to \infty$ and that they are zero in the sampling points on the x-axis, where there are no discontinuities.

In a second test, we apply the filter function D_1^σ on the scalar benchmark signal consisting of a thin plate in the (x, y)-plane, modelled as

$$F(\underline{x}) = \delta(z).$$

Its $\mathbb{H}$-Hermite coefficients

$$F_1^\sigma(\underline{p}) = -\frac{4\pi}{\sigma^2\gamma_1}\exp\left(-\frac{p_3^2}{\sigma^2}\right)p_3e_{12}$$

are zero in the sampling points of the (x, y)-plane and vanish for $p_3 \to \infty$.

A third test is carried out on the scalar benchmark signal

$$R(\underline{x}) = \delta(y)\delta(z)Y(x),$$

where Y stands for the Heaviside step function.
This signal consists of a semi-infinite rod laying along the positive x-axis.
The corresponding $\mathbb{H}$-Hermite coefficients are

$$\begin{aligned} R_1^\sigma(\underline{p}) &= e_1 \left(\frac{2}{\sigma^2 \gamma_1} \exp\left(-\frac{|\underline{p}|^2}{\sigma^2} \right) \right) - \frac{2\sqrt{\pi}}{\sigma^3 \gamma_1} \left(1 + \operatorname{erf}\left(\frac{p_1}{\sigma} \right) \right) \\ & \exp\left(-\frac{p_2^2}{\sigma^2} \right) \exp\left(-\frac{p_3^2}{\sigma^2} \right) (p_2 e_2 + p_3 e_{12}), \end{aligned}$$

with

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

In the sampling points on the x-axis, this means for $p_2 = p_3 = 0$, the $\mathbb{H}$-Hermite coefficients reduce to

$$R_1^\sigma(p_1 e_1) = e_1 \left(\frac{2}{\sigma^2 \gamma_1} \exp\left(-\frac{p_1^2}{\sigma^2} \right) \right).$$

These coefficients have an extreme value at the origin. This demonstrates that the filter function is selective at detecting the discontinuity at the origin. Furthermore, for $p_1 \to -\infty$, independently of p_2 and p_3, the coefficients vanish, while for $p_1 \to +\infty$ they become perpendicular to the x-axis.

Finally we test the filter function on a thin rod of finite length laying on the interval $[-1, 1]$ of the x-axis, modelled as

$$T(\underline{x}) = \delta(y)\delta(z)\chi_{[-1,1]}(x),$$

with $\chi_{[-1,1]}$ the characteristic function on the interval $[-1, 1]$.
Its $\mathbb{H}$-Hermite coefficients are

$$\begin{aligned} T_1^\sigma(\underline{p}) &= e_1 \frac{2}{\sigma^2 \gamma_1} \exp\left(-\frac{p_2^2 + p_3^2}{\sigma^2} \right) \left(\exp\left(-\frac{(1+p_1)^2}{\sigma^2} \right) - \exp\left(-\frac{(1-p_1)^2}{\sigma^2} \right) \right) \\ & -(p_2 e_2 + p_3 e_{12}) \frac{2\sqrt{\pi}}{\sigma^3 \gamma_1} \left(\operatorname{erf}\left(\frac{1+p_1}{\sigma} \right) - \operatorname{erf}\left(\frac{p_1 - 1}{\sigma} \right) \right) \\ & \exp\left(-\frac{p_2^2}{\sigma^2} \right) \exp\left(-\frac{p_3^2}{\sigma^2} \right). \end{aligned}$$

First note that these coefficients vanish, independently from p_2 and p_3, for $p_1 \to \pm\infty$.
For a sampling point $p_1 e_1$ on the x-axis, the $\mathbb{H}$-Hermite coefficients become

$$T_1^\sigma(p_1 e_1) = e_1 \frac{2}{\sigma^2 \gamma_1} \left(\exp\left(-\frac{(1+p_1)^2}{\sigma^2} \right) - \exp\left(-\frac{(1-p_1)^2}{\sigma^2} \right) \right).$$

They achieve a maximum value for $p_1 = 1$ and a minimum value for $p_1 = -1$. So the filter function is again selective at detecting the pointwise singularities.

These four test amply demonstrate that the filter function D_1^σ of the $\mathbb{H}$-Hermite transform is efficient at a pointwise analysis of a signal.

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